

# Helly-type theorem for eigenvectors

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## Abstract

We prove that if any  $\lfloor 3d/2 \rfloor$  or fewer elements of a finite family of linear operators  $\mathbb{K}^d \rightarrow \mathbb{K}^d$  ( $\mathbb{K}$  is an arbitrary field) have a common eigenvector then all operators in the family have a common eigenvector. Moreover,  $\lfloor 3d/2 \rfloor$  cannot be replaced by a smaller number. Also, we study the following problem, achieving partial results: prove that if any  $l = O(d)$  or fewer elements of a finite family of linear operators  $\mathbb{K}^d \rightarrow \mathbb{K}^d$  have a common non-trivial invariant subspace then all operators in the family have a common non-trivial invariant subspace.

## 1 Introduction

We denote the set of numbers  $\{a, a+1, \dots, b\}$  by  $[a, b]$ , where  $a < b$  are positive integer numbers. Also, we use the following notation  $[b] := [1, b]$ .

The well-known Helly's theorem claims.

**Theorem 1.** *Let  $\mathcal{F}$  be a finite family of convex sets in  $\mathbb{R}^d$ . If every  $d+1$  or fewer elements of  $\mathcal{F}$  intersect, then all the sets in the family  $\mathcal{F}$  intersect.*

There is an abundance of literature on Helly-type theorems, see for example [1, 3, 4, 5, 7]. The aim of this work is to answer the following Helly-type question.

**Problem 1.** *Let  $\mathcal{A}$  be a finite family of linear operators  $\mathbb{K}^d \rightarrow \mathbb{K}^d$ , where  $\mathbb{K}$  is an arbitrary field and  $d \geq 2$ . Find minimal  $k = HE(\mathbb{K}^d)$  such that if any  $k$  or fewer elements of  $\mathcal{A}$  have a common eigenvector then there is a common eigenvector for all operators in the family  $\mathcal{A}$ .*

The authors of [2] considered Problem 1 and gave the wrong proof of the fact that  $HE(\mathbb{K}^d) = d+1$  (see Theorem 5.1 in [2]). Our main result is the following theorem.

**Theorem 2.**  $HE(\mathbb{K}^d) = \lfloor 3d/2 \rfloor$  for an arbitrary field  $\mathbb{K}$ .

We prove Theorem 2 in Section 2.

Also, we discuss the following Helly-type question posed by Andrey Voynov [6].

**Problem 2.** *Let  $\mathcal{A}$  be a finite family of linear operators  $\mathbb{K}^d \rightarrow \mathbb{K}^d$ , where  $\mathbb{K}$  is an arbitrary field and  $d \geq 2$ . Find minimal  $l = HI(\mathbb{K}^d)$  such that if any  $l$  or fewer elements of  $\mathcal{A}$  have a common non-trivial invariant subspace then there is a common non-trivial invariant subspace for all operators in the family  $\mathcal{A}$ .*

It is easy to see that  $HI(\mathbb{K}^d) \leq d^2$ . Indeed, suppose that we know that any  $d^2$  or fewer operators in a finite family have a common non-trivial invariant subspace. Note that the space of linear operators is in fact the space of matrices of size  $d \times d$  therefore among operators in the family we can find at most  $d^2$  operators such that any operator in the family is equal to a linear combination of these operators (with coefficients in  $\mathbb{K}$ ). Therefore, because of our assumption all elements of the family have a common non-trivial subspace.

It is not difficult to slightly improve the above estimate  $HI(\mathbb{K}^d) \leq d^2 - d + 1$ . We leave this improvement to reader as an exercise. Moreover, we believe that the following conjecture holds.

**Conjecture 1.**  $HI(\mathbb{K}^d) = O(d)$ .

The following theorem is a partial result confirming Conjecture 1.

**Theorem 3.** *Suppose that  $\mathcal{A}$  is a finite family of linear operators such that  $A_0 \in \mathcal{A}$  have  $d$  eigenvectors in  $\mathbb{K}^d$  with different eigenvalues and any  $2d-1$  or fewer elements of  $\mathcal{A}$  have a common non-trivial invariant subspace then there is a common non-trivial invariant subspace for all elements of  $\mathcal{A}$ .*

Theorem 2 is proved in Section 3.

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## 2 Proof of Theorem 2

The following example shows that  $HE(\mathbb{K}^d) \geq \lfloor 3d/2 \rfloor$ .

*Example.* Let  $\mathbf{e}_1, \dots, \mathbf{e}_{3n}$  be such vectors that

$$\{\mathbf{e}_j : j \in [3n], 3 \nmid j\}$$

is the standard basis vectors of  $\mathbb{K}^{2n}$  and  $\mathbf{e}_{3i} = \mathbf{e}_{3i-2} + \mathbf{e}_{3i-1}$  for any  $i \in [n]$ . Let us introduce the following notations:

$$H_j = \text{span}(\mathbf{e}_i : i \in [3n] \setminus \{j, j + f(j)\}), L_j = \text{span}(\mathbf{e}_j),$$

here  $j \in [3n]$  and

$$f(j) = \begin{cases} 1, & \text{if } j \equiv 1, 2 \pmod{3}; \\ -2, & \text{if } j \equiv 0 \pmod{3}. \end{cases}$$

Let us define a family  $\mathcal{A} = \{A_1, \dots, A_{3n}\}$  of  $3n$  operators. The operator  $A_j$ ,  $j \in [3n]$ , is such that  $H_j$  is the eigenspace associated with the eigenvalue 1 and  $L_j$  is the eigenspace associated with the eigenvalue 0. Obviously,  $A_j$  are well defined and all operators but  $A_j$  have a common eigenvector  $\mathbf{e}_j$ , but all operators do not have a common eigenvector. Analogously, if  $d$  is odd we can construct an example of a family of  $\lfloor 3d/2 \rfloor$  operators  $\mathbb{K}^d \rightarrow \mathbb{K}^d$  such that all operators does not have a common eigenvector and all operators but any operator have.

The proof is based on induction on  $n$ . Suppose that Theorem 2 is shown for families of linear operators containing less then  $n$  operators, where  $n \geq \lfloor 3d/2 \rfloor + 1$ . Let us prove Theorem 2 for a family  $\mathcal{A} = \{A_1, \dots, A_n\}$  of  $n$  operators. By the induction hypothesis, we can find a vector  $\mathbf{v}_i$  that is a common eigenvector of all elements of  $\mathcal{A}$  but  $A_i$ .

Suppose that  $\mathbf{v}_{k+1}, \dots, \mathbf{v}_n$  is the maximal set of linearly independent vectors among  $\mathbf{v}_1, \dots, \mathbf{v}_n$ , then  $n - k \leq d$  and

$$\mathbf{v}_i = \sum_{j=k+1}^n \mu_{i,j} \mathbf{v}_j = \sum_{j \in X_i} \mu_{i,j} \mathbf{v}_j \quad (1)$$

for any  $i \in [k]$ . Here  $X_i$  is the set of  $j \in [k+1, n]$  such that  $\mu_{i,j} \neq 0$ .

**Fact 1.**  $H_i := \text{span}(\{\mathbf{v}_j : j \in X_i\})$  is a common eigenspace for operators  $A_m$ ,  $m \in [n] \setminus (X_i \cup \{i\})$ .

*Proof of Fact 1.* Let us apply the operator  $A_m$  for fixed  $m \in [n] \setminus (X_i \cup \{i\})$  to (1). Because vectors  $\mathbf{v}_j$ ,  $j \in X_i \cup \{i\}$ , are eigenvectors of  $A_m$  thus we get

$$\lambda_{i,m} \mathbf{v}_i = \sum_{j \in X_i} \lambda_{j,m} \mu_{i,j} \mathbf{v}_j,$$

where  $\lambda_{j,m}$ ,  $j \in \{i\} \cup X_i$ , is the eigenvalue corresponding to the operator  $A_m$  and the eigenvector  $\mathbf{v}_j$ . Because vectors  $\mathbf{v}_j$ ,  $j \in X_j \subset [k+1, n]$ , are linearly independent therefore  $\lambda_{j,m} = \lambda$  for all  $j \in \{i\} \cup X_i$ . Thus the operator  $A_m$  is a scalar operator on  $H_i$ .  $\square$

**Fact 2.** Suppose that  $l \in X_i$  for  $i \in [k]$ . Then vectors  $\mathbf{v}_j$ ,  $j \in \{i\} \cup [k+1, n] \setminus \{l\}$ , are linearly independent.

*Proof of Fact 2.* A simple exercise.  $\square$

If  $X_i = \emptyset$  for some  $i \in [k]$  then  $\mathbf{v}_i = \mathbf{0}$ , i.e. we get the contradiction because eigenvectors are non-trivial vectors. If  $X_i = \{m\}$  for some  $i \in [k]$ ,  $m \in [k+1, n]$  then  $\mathbf{v}_i = \mu_{i,m} \mathbf{v}_m$ , i.e.  $\mathbf{v}_i$  is a common eigenvector for all elements of  $\mathcal{A}$ . Therefore,  $|X_i| \geq 2$ , thus without loss of generality we have  $X_{1,2} = X_1 \cap X_2 \neq \emptyset$ . Indeed, otherwise

$$2k \leq \left| \bigcup_{i=1}^k X_i \right| \leq |[k+1, n]| = n - k, \text{ i.e.} \\ n - d \leq k \leq n/3, \quad 2n/3 \leq d,$$

the last inequality contradicts our assumption  $n \geq \lfloor 3d/2 \rfloor + 1$ .

Fix  $l \in X_{1,2} = X_1 \cap X_2 \neq \emptyset$ . Summing (1) for  $i = 1$  and  $i = 2$  with proper coefficients, we get the following equality

$$\begin{aligned} \mu_{2,l} \mathbf{v}_1 - \mu_{1,l} \mathbf{v}_2 = & \sum_{j \in X_1 \setminus X_{1,2}} \mu_{2,l} \mu_{1,j} \mathbf{v}_j - \sum_{j \in X_2 \setminus X_{1,2}} \mu_{1,l} \mu_{2,j} \mathbf{v}_j + \\ & + \sum_{j \in X_{1,2}} (\mu_{2,l} \mu_{1,j} - \mu_{1,l} \mu_{2,j}) \mathbf{v}_j. \end{aligned} \quad (2)$$

Denote by  $X_0$  the set of such  $j \in X_{1,2}$  that  $\mu_{2,l} \mu_{1,j} = \mu_{1,l} \mu_{2,j}$ , i.e. the set of such  $j \in X_{1,2}$  that coefficients corresponding  $\mathbf{v}_j$  in (2) are equal to 0. Note that  $l \in X_0$ . Let us write (2) in the following way

$$\mathbf{v}_1 = \sum_{j \in (\{2\} \cup X_1 \cup X_2) \setminus X_0} \beta_j \mathbf{v}_j, \quad (3)$$

where  $\beta_j \neq 0$  for all  $j \in (\{2\} \cup X_1 \cup X_2) \setminus X_0$ . Applying  $A_l$  to (3) and using that  $\mathbf{v}_j$ ,  $j \in (\{1, 2\} \cup X_1 \cup X_2) \setminus X_0$ , are eigenvectors of  $A_l$  we get

$$\lambda_{1,l} \mathbf{v}_1 = \sum_{j \in (\{2\} \cup X_1 \cup X_2) \setminus X_0} \lambda_{j,l} \beta_j \mathbf{v}_j,$$

where  $\lambda_{j,l}$ ,  $j \in (\{1, 2\} \cup X_1 \cup X_2) \setminus X_0$ , is the eigenvalue corresponding the operator  $A_l$  and  $\mathbf{v}_j$ . By Fact 2 vectors  $\mathbf{v}_j$ ,  $j \in (\{2\} \cup X_1 \cup X_2) \setminus X_0 \subset (\{2\} \cup [k+1, n]) \setminus \{l\}$  are linearly independent therefore we have that  $A_l$  is a scalar operator on  $\text{span}(\{\mathbf{v}_j : j \in (\{1, 2\} \cup X_1 \cup X_2) \setminus X_0\})$  associated with the eigenvalue  $\lambda = \lambda_{1,l}$ , i.e.

$$\begin{aligned} \mathbf{w} &= \mu_{2,l} \mathbf{v}_1 - \sum_{j \in X_1 \setminus X_0} \mu_{2,l} \mu_{1,j} \mathbf{v}_j = \sum_{j \in X_0} \mu_{2,l} \mu_{1,j} \mathbf{v}_j = \\ &= \sum_{j \in X_0} \mu_{1,l} \mu_{2,j} \mathbf{v}_j = \mu_{1,l} \mathbf{v}_2 - \sum_{j \in X_2 \setminus X_0} \mu_{1,l} \mu_{2,j} \mathbf{v}_j \end{aligned}$$

is also an eigenvector of  $A_l$  associated with the eigenvalue  $\lambda$ . Analogously,  $\mathbf{w}$  is an eigenvector of  $A_{l'}$  for every  $l' \in X_0$ . Because of  $\mathbf{w} \in H_1 \cap H_2$  thus by Fact 1  $\mathbf{w}$  is an eigenvector of  $A_j$  for any  $j \in [n] \setminus X_{1,2}$ .

Lastly, let us show that  $\mathbf{w}$  is an eigenvector of any  $A_m$ ,  $m \in X_{1,2} \setminus X_0$ . Again summing (1) for  $i = 1$  and  $i = 2$  with proper coefficients, we have

$$\begin{aligned} \mu_{2,m} \mathbf{v}_1 - \mu_{1,m} \mathbf{v}_2 &= \sum_{j \in X_1 \setminus X_{1,2}} \mu_{2,m} \mu_{1,j} \mathbf{v}_j - \sum_{j \in X_2 \setminus X_{1,2}} \mu_{1,m} \mu_{2,j} \mathbf{v}_j + \\ &\quad + \sum_{j \in X_{1,2}} (\mu_{2,m} \mu_{1,j} - \mu_{1,m} \mu_{2,j}) \mathbf{v}_j. \quad (4) \end{aligned}$$

Denote by  $X'_0$  the set of such  $j \in X_{1,2}$  that  $\mu_{2,m} \mu_{1,j} = \mu_{1,m} \mu_{2,j}$ , i.e. the set of such  $j \in X_{1,2}$  that coefficients corresponding  $\mathbf{v}_j$  in (4) are equal to 0. Note that  $X_0 \cap X'_0 = \emptyset$ , because otherwise we get  $\mu_{2,m} \mu_{1,l'} = \mu_{1,m} \mu_{2,l'}$  and  $\mu_{2,l} \mu_{1,l'} = \mu_{1,l} \mu_{2,l'}$ , where  $l' \in X_0 \cap X'_0$ , thus  $\mu_{2,m} \mu_{1,l} = \mu_{1,m} \mu_{2,l}$ , i.e.  $m \in X_0$ . We now apply the argument used for (2) again, with  $l$  replaced by  $m$ , to obtain that  $A_m$  is a scalar operator on  $\text{span}(\{\mathbf{v}_j : j \in (\{1, 2\} \cup X_1 \cup X_2) \setminus X'_0\})$ , i.e.  $A_m$  is a scalar operator on  $\text{span}(\{\mathbf{v}_j : j \in X_0\})$ . Thus  $\mathbf{w} \in \text{span}(\{\mathbf{v}_j : j \in X_0\})$  is an eigenvector of  $A_m$ .

Therefore,  $\mathbf{w}$  is a common eigenvector of all operators  $A_j$ ,  $j \in [n]$ . Theorem 2 is proved.

### 3 Proof of Theorem 3

We deduce Theorem 3 from the following lemma.

**Lemma 1.** *Given a family  $\mathcal{M} = \{M_1, \dots, M_p\}$  of finite proper and non-empty subsets of  $[q]$ . Suppose that there is no non-empty subset  $I \subset [p]$  such that*

$$\bigcup_{i \in I} M_i = \bigcup_{i \in I \setminus \{j\}} M_i \neq [q]$$

*for any  $j \in I$ . Then  $p \leq 2q - 2$ .*

*Proof of Theorem 3 using Lemma 1.* Suppose that we proved Theorem 3 for any family  $\mathcal{A}$  containing  $A_0$  of size less than  $n$ , where  $n \geq 2d$ . Let us prove Theorem 3 for a family  $\mathcal{A} = \{A_0, A_1, \dots, A_{n-1}\}$ . By induction hypothesis for any  $j \in [n-1]$

there exists an invariant non-trivial subspace  $H_j$  such that it is an invariant for all operators but  $A_j$ .

Denote eigenvectors of  $A_0$  by  $\mathbf{v}_1, \dots, \mathbf{v}_d$  forming a basis of  $\mathbb{K}^d$  because they are associated with different eigenvalues. Therefore, for any non-trivial invariant subspace  $H$  of  $A_0$  there exists  $I \subset [d]$  such that  $H = \text{span}(\mathbf{v}_i : i \in I)$ . Indeed, assume the contrary, i.e.  $H \neq \text{span}(\mathbf{v}_i, i \in I)$  for any  $I \subset [d]$ , i.e. there are vectors in  $H$  such that they could be represented as sums of eigenvectors of  $A_0$  that do not belong to  $H$ . Choose such a vector  $\mathbf{v}$  that it has the minimal number of terms in its representation, i.e.

$$\mathbf{v} = \sum_{i=1}^k \alpha_i \mathbf{v}_{l_i}, \quad (5)$$

where  $\mathbf{v}_{l_i} \notin H$ ,  $\alpha_i \in \mathbb{K}, \alpha_i \neq 0, i \in [k]$ , and  $k > 1$  is minimal. Applying  $A_0$  to (5) we get that

$$\mathbf{v}' = \sum_{i=1}^k \alpha_i \lambda_{l_i} \mathbf{v}_{l_i} \in H, \quad (6)$$

where  $\lambda_{l_i}, i \in [k]$ , is the eigenvalue of  $A_0$  corresponding to the eigenvector  $\mathbf{v}_{l_i}$ . Therefore, we have that the vector

$$\mathbf{v}' - \lambda_{l_k} \mathbf{v} = \sum_{i=1}^{k-1} \alpha_i (\lambda_{l_i} - \lambda_{l_k}) \mathbf{v}_{l_i} \in H$$

can be represented as the sum of  $k-1$  eigenvectors of  $A_0$  that do not belong to  $H$ . This contradiction shows that for each  $i, i \in [n-1]$ , we can assign  $M_i \subset [d]$  such that  $H_i = \text{span}(\mathbf{v}_i : i \in M_i)$  because  $H_i$  are invariant subspaces of  $A_0$ . By Lemma 1 there exists a non-empty set  $I \subset [d]$  such that

$$M = \bigcup_{i \in I} M_i = \bigcup_{i \in I \setminus \{j\}} M_i \neq [d]$$

for any  $j \in I$ , i.e.  $\text{span}(\mathbf{v}_j : j \in M) \neq \mathbb{K}^d$  is a common non-trivial invariant subspace of all operators in the family  $\mathcal{A}$ .  $\square$

*Proof of Lemma 1.* The proof is by induction on  $q$ . Lemma 1 for  $q = 1$  is trivial. Suppose that Lemma 1 is proved for  $q < n$ , let us prove it for  $q = n$ . If there are two sets  $M_i, M_j \in \mathcal{M}$  such that  $M_i \not\subset M_j, M_j \not\subset M_i$  and  $M_i \cup M_j \neq [q]$  then we will replace (in  $\mathcal{M}$ )  $M_j$  by  $M_i \cup M_j$  (we do not change  $M_i$ ). It is easy to see that the changed  $\mathcal{M}$  satisfies conditions of Lemma 1. We make such replacements till it is possible to do them. Obviously, we can not do this operation infinitely many times because after each replacement the total cardinality (taken with multiplicity) of sets increases. Finally, we get such  $\mathcal{M}$  that for every two elements of  $\mathcal{M}$  one contains another or their union is  $[q]$ . Consider a set  $A = \{a_1, \dots, a_k\} \in \mathcal{M}$  such that it does not contain another element of  $\mathcal{M}$  as its subset. Suppose that  $K_1, \dots, K_l \in \mathcal{M}$  are sets that do not contain  $A$  then  $l \leq 2k-1$ . Indeed,  $X = [q] \setminus A \subset K_i$  therefore each  $K_i$  is defined by  $K_i \cap A$ . There is at most one such  $m \in [l]$  such that  $K_m \cap A = \emptyset$ . Note that the family  $\{K_i \cap A : i \in [l] \setminus \{m\}\}$  satisfies conditions of Lemma 1. By the induction hypothesis  $l-1 \leq 2k-2$ , i.e.  $l \leq 2k-1$ . If we delete  $A$  and  $K_i$  from  $\mathcal{M}$  (i.e. at most  $2k$  elements) and delete  $a_1, \dots, a_k$  in other elements of  $\mathcal{M}$  then the new family will satisfy conditions of Lemma 1, i.e. it has at most  $2(q-k)-2$  elements. Thus the original family  $\mathcal{M}$  contains at most  $2k + 2(q-k) - 2 = 2q - 2$  elements.  $\square$

*Example.* Obviously, the following example shows that the inequality in Lemma 1 is tight:

$$\mathcal{M} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \dots, \{1, \dots, q-1\}\} \cup \\ \cup \{\{q\}, \{q, q-1\}, \{q, q-1, q-2\}, \dots, \{q, \dots, 2\}\}.$$

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